Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Computational and Applied Mathematics 179 (2005) 157–173
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A Christoffel–Darboux formula and a Favard's theorem for orthogonal Laurent polynomials on the unit circle ☆

Ruymán Cruz-Barroso*, Pablo González-Vera

Department of Mathematical Analysis, La Laguna University, 38271 La Laguna, Tenerife, Spain

Received 13 November 2003

Dedicated to professor Olav Njåstad on the occasion of his 70th birthday

Abstract

Let $\{\varphi_k(z)\}_{k=0}^{\infty}$ be the family of orthonormal Laurent polynomials on the unit circle which spans Λ in the "ordering" induced by $p(n)=E[(n+1)/2]$. From the three-term recurrence relation satisfied by $\{\varphi_k(z)\}_{k=0}^{\infty}$ we deduce a Christoffel–Darboux formula. Particular examples are considered and a Favard-type theorem is proved. A connection with the ordering induced by $p(n)=E[n/2]$ is also established.

Keywords: Orthogonal Laurent polynomials on the unit circle; Three-term recurrence relations; Christoffel–Darboux formula; Favard's theorem

1. Introduction

In this paper, we shall be concerned with a positive Borel measure μ supported on the unit circle $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$ and with functions like $L(z)=\sum_{j=p}^{q}\alpha_j z^j$ where $p,q\in\mathbb{Z}$ $(p\leqslant q)$ and $\alpha_j\in\mathbb{C}$ for all $j=p,\ldots,q$ (Laurent polynomials). We set $\Lambda_{p,q}=\{L(z)=\sum_{j=p}^{q}\alpha_j z^j:\alpha_j\in\mathbb{C}\}$ and Λ the space of all Laurent polynomials. We also denote by Π_n the space of all (standard) polynomials of degree at most n and by Π the space of all polynomials (observe that $\Lambda_{0,n}=\Pi_n$ for all $n=0,1,2,\ldots$).

☆ This work is partially supported by the research project BFM 2001-3411 of the Spanish Ministry of Science and Technology and by the research project PI-2002-023 of Gobierno de Canarias.
* Corresponding author.
E-mail address: szegoquadrature@hotmail.com (R. Cruz-Barroso).

0377-0427/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2004.09.039

Laurent polynomials can be considered as the basic ingredient when dealing with approximation questions on the unit circle (recall that any continuous function can be uniformly approximated by Laurent polynomials, see [15]), as Szegö quadratures, Trigonometric Moment Problem, Carathéodory–Féjer interpolation problem and so on. Hence, as indicated in [13] it becomes interesting to find sequences of Laurent polynomials which are both orthogonal and span Δ. Here, orthogonality should be understood with respect to the inner product associated with the measure μ, i.e.,

$$\langle f, g\rangle_\mu = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\mu(\theta) = \int_{\mathbb{T}} f(z)\overline{g(z)}\,\mathrm{d}\mu(z), \quad f, g \in L_2^\mu(\mathbb{T}). \tag{1.1}$$

Furthermore, a crucial property should be remarked: the space Π is dense in $L_2^\mu(\mathbb{T})$ (with respect to the L_2^μ-norm) if and only if μ does not fulfil the Szegö's condition, i.e., $\int_{\mathbb{T}} \ln\, \mu'(\theta)\,\mathrm{d}\theta = -\infty$. However, Δ is dense in $L_2^\mu(\mathbb{T})$ independently of this condition (see e.g., [2]). Here it should be taken into account that a great number of interesting cases, e.g., that the Lebesgue measure, fulfils this condition. Thus, one sees that Laurent polynomials are also needed to obtain a basis of $L_2^\mu(\mathbb{T})$ (see [11] and compare with [13]). For this purpose and in order to generate a family of nested subspaces of Laurent polynomials similar to $\{\Pi_k\}_{k\geqslant 0}$ in the standard polynomial case, we will start from two nondecreasing sequences of nonnegative integers $\{p(n)\}_{n=0}^\infty$ and $\{q(n)\}_{n=0}^\infty$ such that $p(n)+q(n)=n$ for all $n=0,1,2,\ldots$ and set

$$\mathscr{L}_n = \Delta_{-p(n),q(n)} = \mathrm{span}\{z^j : -p(n)\leqslant j\leqslant q(n)\}.$$

From this construction it holds that $\mathscr{L}_0 = \mathrm{span}\{1\}$, $\dim(\mathscr{L}_n) = n+1$ and $\mathscr{L}_n \subset \mathscr{L}_{n+1}$. When one wishes to guarantee that $\bigcup_{n=0}^\infty \mathscr{L}_n = \Delta$ we must assume that $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} q(n) = \infty$. If $p(n)=0$, then $q(n)=n$ and $\mathscr{L}_n = \Delta_{0,n} = \Pi_n$.

If $\{\tau_k(z)\}_{k=0}^n$ is a system of linearly independent functions in $\mathscr{L}_n$ (e.g., $\{z^{-p(n)},\ldots,z^{q(n)}\}$), by applying the Gram–Schmidt orthogonalization process a new system of linearly independent functions $\{\psi_k(z)\}_{k=0}^n$ can be obtained such that $\psi_n(z)$ is a linear combination of the $n+1$ functions $\{\tau_k(z)\}_{k=0}^n$ and

$$\langle\psi_n(z),\psi_m(z)\rangle_\mu = \int_{\mathbb{T}} \psi_n(z)\overline{\psi_m(z)}\,\mathrm{d}\mu(z) = k_n\delta_{n,m}, \quad k_n>0, \quad \delta_{n,m} = \begin{cases} 0 & \text{if } n\neq m, \\ 1 & \text{if } n=m. \end{cases}$$

When the process is repeated for each natural n, an essentially unique system $\{\psi_n(z)\}_{n=0}^\infty$ of orthogonal functions with respect to the measure μ and the "ordering" induced by $\{p(n)\}_{n=0}^\infty$ is obtained.

In the rest of the paper we will take $p(n) = E[(n+1)/2]$, where, as usual, $E[x]$ denotes the integer part of x (and hence, $q(n) = n - p(n) = E[n/2]$). This sequence induces the following "ordering" in Δ

$$\mathscr{L}_0 = \Delta_{0,0}, \quad \mathscr{L}_1 = \Delta_{-1,0}, \quad \mathscr{L}_2 = \Delta_{-1,1}, \quad \mathscr{L}_3 = \Delta_{-2,1}, \quad \mathscr{L}_4 = \Delta_{-2,2}, \ldots . \tag{1.2}$$

We will denote by $\{\phi_n(z)\}_{n=0}^\infty$ the corresponding sequence of monic orthogonal Laurent polynomials on the unit circle with respect to the measure μ and the above ordering. This family satisfies

1. $\phi_n \in \mathscr{L}_n\backslash\mathscr{L}_{n-1}$, $n=1,2,\ldots$.
2. $\langle\phi_n,\phi_m\rangle_\mu = k_n\delta_{n,m}$, $k_n>0$

and the leading coefficients (coefficient of $z^{-(n+1)}$ in $\phi_{2n+1}(z)$ and z^n in $\phi_{2n}(z)$) are equal to 1. We will also denote by $\{\varphi_n(z)\}_{n=0}^\infty$ the corresponding orthonormal sequence, i.e., when $k_n=1$ in 2. for $n=1,2,\ldots$.

The paper is organized as follows. In Section 2 a connection between the Laurent polynomials $\phi_n(z)$ ($\varphi_n(z)$) and the monic orthogonal (orthonormal) Szegö polynomials is given. From the three-term recurrence relation satisfied by the family $\{\phi_k(z)\}_{k=0}^{\infty}$ a similar one for the family $\{\varphi_k(z)\}_{k=0}^{\infty}$ is deduced. A connection between the family $\{\phi_k(z)\}_{k=0}^{\infty}$ ($\{\varphi_k(z)\}_{k=0}^{\infty}$) and the family of Laurent polynomials $\{\tilde{\phi}_k(z)\}_{k=0}^{\infty}$ ($\{\tilde{\varphi}_k(z)\}_{k=0}^{\infty}$) obtained from the nested sequence of Laurent polynomials $\tilde{\mathscr{L}}_n = \Lambda_{-q(n),p(n)}$ is also given. The main results are established in the two following sections. In Section 3 a Christoffel–Darboux formula for the family $\{\varphi_k(z)\}_{k=0}^{\infty}$ and a converse result are proved. This formula is illustrated with some particular examples: Lebesgue measure and rational modifications of the Lebesgue measure. Finally, in Section 4 a Favard's theorem is also obtained. The two topics studied in Sections 3 and 4 have been previously considered in the framework of rational orthogonal functions (see Chapters 3 and 8 of monograph [2]) as well as in the framework of orthogonal polynomials on the unit circle (see e.g. [7]).

2. Preliminary results

Let $\{\rho_n(z)\}_{n=0}^{\infty}$ be the family of (monic) Szegö polynomials (orthogonal with respect to the inner product (1.1)), where we set $\rho_n(z) = z^n + \cdots + \delta_n$ for all $n = 0, 1, 2, \ldots$. The sequence $\{\delta_n\}_{n=0}^{\infty}$ is called the sequence of *Schur–Szegö parameters* (*reflection coefficients* or *Verblunsky parameters*) for the measure μ. It is known that $|\delta_n| < 1$ for all $n = 1, 2, \ldots$. We will also denote by $\omega_n(z) = \gamma_n \rho_n(z)$ the nth orthonormal Szegö polynomial, where

$$\gamma_n = \frac{1}{\|\rho_n(z)\|_\mu} = \frac{1}{\sqrt{\langle \rho_n(z), \rho_n(z)\rangle_\mu}} > 0.$$

Consider also the so-called reversed polynomials $\rho_n^*(z) := z^n \rho_{n*}(z)$, where in general, for a given function $f(z)$, the sub-star conjugation is defined as $f_*(z) = \overline{f(1/\bar{z})}$. Hence, $\rho_n^*(z) = z^n \overline{\rho_n(1/\bar{z})} = \overline{\delta_n} z^n + \cdots + 1$ and $\omega_n^*(z) := z^n \overline{\omega_n(1/\bar{z})} = \gamma_n \overline{\delta_n} z^n + \cdots + \gamma_n$ (nth reversed monic and orthonormal Szegö polynomial, respectively).

From the orthogonality conditions the following relation between $\phi_n(z)$ and $\rho_n(z)$ is deduced (see [4,13]).

Lemma 2.1. *Let $\{\phi_n(z)\}_{n=0}^{\infty}$ be a sequence of monic Laurent polynomials so that $\phi_n(z) = N_n(z)/z^{p(n)}$ with $N_n(z) \in \Pi_n$ for $n = 0, 1, 2, \ldots$ and $p(n) = E[(n+1)/2]$. Then $\{\phi_n(z)\}_{n=0}^{\infty}$ is the sequence of monic orthogonal Laurent polynomials if and only if $\{\rho_n(z)\}_{n=0}^{\infty}$ given by*

$$\rho_{2n}(z) = N_{2n}(z), \quad \rho_{2n+1}(z) = N_{2n+1}^*(z), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

is the corresponding sequence of monic Szegö polynomials.

Remark 2.2. Observe that this property has nothing to do with what happens when considering orthogonal Laurent polynomials on the real line (see e.g. [10,4]).

Hence it follows that

$$\|\phi_n(z)\|_\mu = \|\rho_n(z)\|_\mu = \|\rho_n^*(z)\|_\mu. \tag{2.2}$$

From (2.1) and (2.2) a relation between $\varphi_n(z)$ and $\omega_n(z)$ is also deduced:

$$\varphi_{2k}(z) = \frac{\phi_{2k}(z)}{\|\phi_{2k}(z)\|_\mu} = \frac{\rho_{2k}(z)}{\|\rho_{2k}(z)\|_\mu}\frac{1}{z^k} = \frac{\omega_{2k}(z)}{z^k}, \tag{2.3}$$

$$\varphi_{2k+1}(z) = \frac{\phi_{2k+1}(z)}{\|\phi_{2k+1}(z)\|_\mu} = \frac{\rho^*_{2k+1}(z)}{\|\rho^*_{2k+1}(z)\|_\mu}\frac{1}{z^{k+1}} = \frac{\omega^*_{2k+1}(z)}{z^{k+1}}. \tag{2.4}$$

Now, we introduce the sequence of nonnegative real numbers $\{\eta_n\}_{n=0}^\infty$ defined by

$$\eta_n = +\sqrt{1-|\delta_n|^2} \in (0,1], \quad n = 0,1,2,\ldots\ . \tag{2.5}$$

From [14] it holds for all $n = 1, 2, 3, \ldots$ the relation $(\gamma_{n-1}/\gamma_n)^2 = \eta_n^2$. Since $\eta_n \in (0,1]$ and $\gamma_n > 0$ for all $n = 0, 1, 2, \ldots$, then

$$\eta_n = \frac{\gamma_{n-1}}{\gamma_n}, \quad n = 1,2,3,\ldots\ . \tag{2.6}$$

It is known (see [13,4] for different proofs based upon certain continued fractions and the recurrence relation satisfied by the Szegö polynomials, respectively) that the sequence of monic orthogonal Laurent polynomials $\{\phi_n(z)\}_{n=0}^\infty$ satisfies the three-term recurrence relation

$$\phi_n(z) = (A_n + \overline{A_{n-1}}z^{(-1)^n})\phi_{n-1}(z) + \eta_{n-1}^2 z^{(-1)^n}\phi_{n-2}(z), \quad n \geqslant 2, \tag{2.7}$$

where

$$\phi_0(z) \equiv 1, \quad \phi_1(z) = \overline{\delta_1} + \frac{1}{z}, \quad A_n = \begin{cases} \overline{\delta_n} & \text{if } n \text{ is even}, \\ \delta_n & \text{if } n \text{ is odd}. \end{cases} \tag{2.8}$$

Now, taking into account

$$\|\rho_n(z)\|_\mu^2 = \langle \rho_n(z), \rho_n(z)\rangle = \frac{\Gamma_n}{\Gamma_{n-1}} = \eta_n^2 \frac{\Gamma_{n-1}}{\Gamma_{n-2}}, \tag{2.9}$$

where Γ_n is the nth Toeplitz determinant for the trigonometric moments $\mu_k = \int_{\mathbb{T}} z^{-k}\,\mathrm{d}\mu(z)$, $k = 0, \pm 1, \pm 2, \ldots$ and making use of (2.2) and (2.9), from (2.7) we can easily deduce the following.

Lemma 2.3. *The family of orthonormal Laurent polynomials* $\{\varphi_k(z)\}_{k=0}^\infty$ *satisfies the three-term recurrence relation*

$$\eta_n\varphi_n(z) = (A_n + \overline{A_{n-1}}z^{(-1)^n})\varphi_{n-1}(z) + \eta_{n-1}z^{(-1)^n}\varphi_{n-2}(z), \quad n \geqslant 2, \tag{2.10}$$

where

$$\varphi_i(z) = \frac{\phi_i(z)}{\|\phi_i(z)\|_\mu} \quad i = 0, 1$$

and η_n *and* A_n *are given by* (2.5) *and* (2.8), *respectively*.

Remark 2.4. In the proofs of the two following sections relation (2.10) will be needed to be expressed for $k \geqslant 1$ as

$$\eta_{2k}\varphi_{2k}(z) = (\delta_{2k} + \delta_{2k-1}z)\varphi_{2k-1}(z) + \eta_{2k-1}z\varphi_{2k-2}(z)$$

and

$$\eta_{2k+1}\varphi_{2k+1}(z)=\left(\overline{\delta_{2k+1}}+\overline{\delta_{2k}}\,\frac{1}{z}\right)\varphi_{2k}(z)+\eta_{2k}\,\frac{1}{z}\,\varphi_{2k-1}(z)$$

or, equivalently,

$$\eta_{2k+1}\varphi_{2k}(z)=-\delta_{2k+1}\varphi_{2k+1}(z)+\frac{1}{z}(\eta_{2k+2}\varphi_{2k+2}(z)-\delta_{2k+2}\varphi_{2k+1}(z)) \tag{2.11}$$

and

$$\eta_{2k}\varphi_{2k-1}(z)=-\overline{\delta_{2k}}\varphi_{2k}(z)+z(\eta_{2k+1}\varphi_{2k+1}(z)-\overline{\delta_{2k+1}}\varphi_{2k}(z)). \tag{2.12}$$

We conclude this section by giving a connection between $\{\phi_k(z)\}_{k=0}^{\infty}$ and the family of orthogonal Laurent polynomials $\{\tilde{\phi}_k(z)\}_{k=0}^{\infty}$ with respect to the measure μ and the ordering induced by the sequences

$$\tilde{p}(n)=E\left[\frac{n}{2}\right],\quad \tilde{q}(n)=n-\tilde{p}(n)=E\left[\frac{n+1}{2}\right],\ \ n=0,1,\ldots\ . \tag{2.13}$$

Indeed, by setting $\mathscr{L}_{n*}=\{f\in\Lambda : f_*\in\mathscr{L}_n\}$, then it can be easily checked that $\tilde{\mathscr{L}}_n=\Lambda_{-\tilde{p}(n),\tilde{q}(n)}=\mathscr{L}_{n*}$. Therefore, if $\{\tilde{\phi}_k(z)\}_{k=0}^{\infty}$ and $\{\tilde{\varphi}_k(z)\}_{k=0}^{\infty}$ denotes the corresponding sequences of monic and orthonormal Laurent polynomials induced by the ordering (2.13), then $\tilde{\phi}_n(z)=\phi_{n*}(z)$ and $\tilde{\varphi}_n(z)=\varphi_{n*}(z)$ for $n=0,1,2,\ldots$. As a consequence, taking sub-star conjugation in both hand sides of equalities (2.7)–(2.8) and (2.10) we deduce the following.

Corollary 2.5. *The family of monic orthogonal Laurent polynomials $\{\tilde{\phi}_k(z)\}_{k=0}^{\infty}$ satisfies the three-term recurrence relation*

$$\tilde{\phi}_n(z)=(\overline{A_n}+A_{n-1}z^{(-1)^{n+1}})\tilde{\phi}_{n-1}(z)+\eta_{n-1}^2z^{(-1)^{n+1}}\tilde{\phi}_{n-2}(z),\quad n\geqslant 2, \tag{2.14}$$

where

$$\tilde{\phi}_0(z)\equiv 1,\quad \tilde{\phi}_1(z)=\delta_1+z,\quad A_n=\begin{cases}\delta_n & \text{if } n \text{ is even},\\ \overline{\delta_n} & \text{if } n \text{ is odd}.\end{cases} \tag{2.15}$$

Corollary 2.6. *The family of orthonormal Laurent polynomials $\{\tilde{\varphi}_k(z)\}_{k=0}^{\infty}$ satisfies the three-term recurrence relation*

$$\eta_n\tilde{\varphi}_n(z)=(\overline{A_n}+A_{n-1}z^{(-1)^{n+1}})\tilde{\varphi}_{n-1}(z)+\eta_{n-1}z^{(-1)^{n+1}}\tilde{\varphi}_{n-2}(z),\quad n\geqslant 2, \tag{2.16}$$

where

$$\tilde{\varphi}_i(z)=\frac{\tilde{\phi}_i(z)}{\|\tilde{\phi}_i(z)\|_{\mu}}\ i=0,1,\quad A_n=\begin{cases}\delta_n & \text{if } n \text{ is even},\\ \overline{\delta_n} & \text{if } n \text{ is odd}.\end{cases} \tag{2.17}$$

Remark 2.7. In [13] Thron considered the families $\{\phi_k(z)\}_{k=0}^{\infty}$ and $\{\tilde{\phi}_k(z)\}_{k=0}^{\infty}$ separately and proved relations (2.7)–(2.8) and (2.14)–(2.15) considering two certain continued fractions. From the above, taking $*$-operation (since it is an involution) in both sides of the three-term recurrence relation for one of such families, the three-term recurrence relation for the other family is automatically deduced. On the other hand, in [11] biorthogonal systems of trigonometric polynomials were introduced giving rise to orthogonal systems of Laurent polynomials with respect to the ordering induced by $p(n) = n/2$; n even, i.e., $\Delta_{0,0}, \Delta_{-1,1}, \Delta_{-2,2}, \Delta_{-3,3}, \Delta_{-4,4}, \ldots$.

3. Christoffel–Darboux formula

In this section we will give a simple expression of the reproducing kernel function for $\mathscr{L}_n, n=0, 1, 2, \ldots,$ similar to the Christoffel–Darboux formula for the standard polynomial case (that is, $p(n) = 0$ for all $n = 0, 1, 2, \ldots$) (see [6]). Recall that the reproducing kernel function for $\mathscr{L}_n$, namely

$$\mathscr{K}_n(z,\xi) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\xi)} \tag{3.1}$$

takes its name since it satisfies the property

$$L(z) = \langle L(\xi), \mathscr{K}_n(\xi, z)\rangle_\mu \quad \text{for all } L(z) \in \mathscr{L}_n. \tag{3.2}$$

Theorem 3.1 (*Christoffel–Darboux formula*). *If* $\{\varphi_k(z)\}_{k=0}^{\infty}$ *denotes the family of orthonormal Laurent polynomials and* $z, \xi \in \mathbb{C}\backslash\{0\}$ *then*

$$\begin{aligned}\mathscr{K}_n(z,\xi) &= \eta_{n+1}(-z\overline{\xi})^{d(n)}\\ &\quad\times \frac{(\eta_{n+1}\overline{\varphi_n(\xi)} + A_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_n(z) + (\overline{A_{n+1}}\,\overline{\varphi_n(\xi)} - \eta_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_{n+1}(z)}{1 - z\overline{\xi}},\end{aligned} \tag{3.3}$$

where

$$d(n) = \frac{(-1)^n + 1}{2}, \quad A_n = \begin{cases} \overline{\delta_n} & \text{if } n \text{ is even},\\ \delta_n & \text{if } n \text{ is odd},\end{cases} \tag{3.4}$$

$\{\delta_n\}_{n=0}^{\infty}$ *being the* reflection coefficients *for the measure* μ *and* $\{\eta_n\}_{n=0}^{\infty}$ *given by* (2.5).

Proof. Recall that $\mathscr{L}_n = \text{span}\{z^j : -p(n) \leqslant j \leqslant q(n)\} = \Delta_{-p(n),q(n)}$ being $p(n) = E[(n+1)/2]$, $q(n) = n - p(n) = E[n/2]$ and the fundamental property of the reproducing kernel:

$$\begin{aligned}\langle R(z), \mathscr{K}_n(z,\xi)\rangle &= \int_{\mathbb{T}} R(z)\overline{\mathscr{K}_n(z,\xi)}\,\mathrm{d}\mu(z) = \int_{\mathbb{T}} R(z) \sum_{k=0}^{n} \overline{\varphi_k(z)}\varphi_k(\xi)\,\mathrm{d}\mu(z)\\ &= \int_{\mathbb{T}} R(z)\mathscr{K}_n(\xi,z)\,\mathrm{d}\mu(z) = R(\xi), \quad \forall R \in \mathscr{L}_n.\end{aligned} \tag{3.5}$$

We distinguish two situations.

Case $n = 2k$: First of all, observe that

$$\mathscr{L}_n = \Delta_{-k,k}, \quad \mathscr{L}_{n-1} = \Delta_{-k,k-1}, \quad \mathscr{L}_{n+1} = \Delta_{-(k+1),k}.$$

If we take $M(t) \in \mathscr{L}_{n-1}$, then $(\xi - t)M(t) \in \Delta_{-k,k} = \mathscr{L}_{2k} = \mathscr{L}_n$. Setting $R(t) = (\xi - t)M(t) \in \mathscr{L}_n$, then $\int_{\mathbb{T}} R(z)\mathscr{K}_n(\omega, z)\,\mathrm{d}\mu(z) = R(\omega)$ and, when $\omega = \xi$ we obtain $\int_{\mathbb{T}} R(z)\mathscr{K}_n(\xi, z)\,\mathrm{d}\mu(z) = R(\xi) = 0$. Hence,

$$\int_{\mathbb{T}} M(z)(\xi - z)\mathscr{K}_n(\xi, z)\,\mathrm{d}\mu(z) = 0 \Rightarrow \int_{\mathbb{T}} M(z)(\xi - z)\overline{\mathscr{K}_n(z, \xi)}\,\mathrm{d}\mu(z) = 0$$
$$\Rightarrow \int_{\mathbb{T}} M(z)\overline{(\overline{\xi} - \overline{z})\mathscr{K}_n(z, \xi)}\,\mathrm{d}\mu(z) = 0$$

and since $z \in \mathbb{T}$, $\overline{z} = 1/z$, so

$$\int_{\mathbb{T}} M(z)\overline{\left(\overline{\xi} - \frac{1}{z}\right)\mathscr{K}_n(z, \xi)}\,\mathrm{d}\mu(z) = 0$$

yielding $(\overline{\xi} - 1/z)\mathscr{K}_n(z, \xi) \perp \mathscr{L}_{n-1}$, that is,

$$\left\langle M(z), \left(\overline{\xi} - \frac{1}{z}\right)\mathscr{K}_n(z, \xi)\right\rangle = 0, \quad \forall M \in \mathscr{L}_{n-1}. \tag{3.6}$$

Now, $\overline{\xi}\mathscr{K}_n(z, \xi) \in \Lambda_{-k,k} = \mathscr{L}_n \subset \mathscr{L}_{n+1}$ and $(1/z)\mathscr{K}_n(z, \xi) \in \Lambda_{-(k+1),k-1} \subset \mathscr{L}_{n+1}$. Then, $(\overline{\xi} - 1/z)\mathscr{K}_n(z, \xi) \in \mathscr{L}_{n+1}$ and one can write

$$\left(\overline{\xi} - \frac{1}{z}\right)\mathscr{K}_n(z, \xi) = \sum_{k=0}^{n+1} a_k(\xi)\varphi_k(z).$$

By (3.6) it holds $a_k(\xi) \equiv 0$ for all $0 \leqslant k \leqslant n-1$, so

$$\left(\overline{\xi} - \frac{1}{z}\right)\mathscr{K}_n(z, \xi) = a_n(\xi)\varphi_n(z) + a_{n+1}(\xi)\varphi_{n+1}(z)$$
$$\Rightarrow \frac{z\overline{\xi} - 1}{z}\sum_{k=0}^{n}\varphi_k(z)\overline{\varphi_k(\xi)} = a_n(\xi)\varphi_n(z) + a_{n+1}(\xi)\varphi_{n+1}(z)$$

or equivalently (setting $a_{n+1}(\xi) = b_n(\xi)$),

$$\mathscr{K}_n(z, \xi)(1 - z\overline{\xi}) = (1 - z\overline{\xi})\sum_{k=0}^{n}\varphi_k(z)\overline{\varphi_k(\xi)} = -z(a_n(\xi)\varphi_n(z) + b_n(\xi)\varphi_{n+1}(z)). \tag{3.7}$$

Now, in this case it holds that

$$\varphi_{n-1}(z) = \frac{\omega^*_{n-1}(z)}{z^k} = \frac{\gamma_{2k-1}\overline{\delta_{2k-1}}z^{2k-1} + \cdots + \gamma_{2k-1}}{z^k} \in \Lambda_{-k,k-1},$$
$$\varphi_n(z) = \frac{\omega_n(z)}{z^k} = \frac{\gamma_{2k}z^{2k} + \cdots + \gamma_{2k}\delta_{2k}}{z^k} \in \Lambda_{-k,k},$$
$$\varphi_{n+1}(z) = \frac{\omega^*_{n+1}(z)}{z^{k+1}} = \frac{\gamma_{2k+1}\overline{\delta_{2k+1}}z^{2k+1} + \cdots + \gamma_{2k+1}}{z^{k+1}} \in \Lambda_{-(k+1),k}. \tag{3.8}$$

If we compare the coefficients of z^{k+1} and z^{-k} in both sides of (3.7) we obtain the relations

$$\gamma_n \overline{\xi\varphi_n(\xi)} = \gamma_n a_n(\xi) + \gamma_{n+1}\overline{\delta_{n+1}} b_n(\xi)$$

$$\gamma_{n-1}\overline{\varphi_{n-1}(\xi)} + \gamma_n \delta_n \overline{\varphi_n(\xi)} = -\gamma_{n+1} b_n(\xi),$$

respectively. From (2.6) these expressions imply

$$b_n(\xi) = -\eta_{n+1}(\eta_n \overline{\varphi_{n-1}(\xi)} + \delta_n \overline{\varphi_n(\xi)}), \tag{3.9}$$

$$a_n(\xi) = \overline{\xi\varphi_n(\xi)} - \frac{\overline{\delta_{n+1}}}{\eta_{n+1}} b_n(\xi). \tag{3.10}$$

Now from (2.12) and (3.9) an expression for $b_{2k}(\xi)$ in terms of $\varphi_{2k}(\xi)$ and $\varphi_{2k+1}(\xi)$ follows

$$b_{2k}(\xi) = \eta_{2k+1}\bar{\xi}(\delta_{2k+1}\overline{\varphi_{2k}(\xi)} - \eta_{2k+1}\overline{\varphi_{2k+1}(\xi)}) \tag{3.11}$$

and hence from (3.10) it also holds that

$$a_{2k}(\xi) = \eta_{2k+1}\bar{\xi}(\overline{\delta_{2k+1}\varphi_{2k+1}(\xi)} + \eta_{2k+1}\overline{\varphi_{2k}(\xi)}). \tag{3.12}$$

If we combine (3.7), (3.11) and (3.12) then we deduce the Christoffel–Darboux formula for the even case obtaining

$$\begin{aligned}\mathscr{K}_n(z,\xi)(1-z\bar{\xi}) = \eta_{n+1}(-z\bar{\xi})[&(\eta_{n+1}\overline{\varphi_n(\xi)} + \overline{\delta_{n+1}\varphi_{n+1}(\xi)})\varphi_n(z) \\ &+ (\delta_{n+1}\overline{\varphi_n(\xi)} - \eta_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_{n+1}(z)].\end{aligned} \tag{3.13}$$

Case $n = 2k+1$: Proceeding in an analogous way to the even case, we can write

$$\mathscr{K}_n(z,\xi)(1-z\bar{\xi}) = (1-z\bar{\xi})\sum_{k=0}^{n}\varphi_k(z)\overline{\varphi_k(\xi)} = \bar{\xi}(a_n(\xi)\varphi_n(z) + b_n(\xi)\varphi_{n+1}(z)), \tag{3.14}$$

where

$$a_n(\xi) = \eta_{n+1}\frac{1}{\bar{\xi}}(\eta_{n+1}\overline{\varphi_n(\xi)} + \delta_{n+1}\overline{\varphi_{n+1}(\xi)}) \tag{3.15}$$

and

$$b_n(\xi) = \eta_{n+1}\frac{1}{\bar{\xi}}(\overline{\delta_{n+1}\varphi_n(\xi)} - \eta_{n+1}\overline{\varphi_{n+1}(\xi)}). \tag{3.16}$$

Thus, from (3.14)–(3.16) we also deduce the Christoffel–Darboux formula for the odd case obtaining

$$\begin{aligned}&\mathscr{K}_n(z,\xi)(1-z\bar{\xi}) \\ &\quad = \eta_{n+1}[(\eta_{n+1}\overline{\varphi_n(\xi)} + \delta_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_n(z) + (\overline{\delta_{n+1}\varphi_n(\xi)} - \eta_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_{n+1}(z)].\end{aligned} \tag{3.17}$$

Finally, from (3.13) and (3.17) the Christoffel–Darboux formula (3.3) is deduced. □

In order to illustrate the above formula we will restrict ourselves to some examples of absolutely continuous measures μ, i.e., $\mathrm{d}\mu(\theta) = \omega(\theta)\,\mathrm{d}\theta$, $\omega(\theta) > 0$ a.e. on $[-\pi, \pi]$ (weight function) considering the following.

Example 3.2 (*Lebesgue measure*). Consider $\mathrm{d}\mu(\theta) = \mathrm{d}\theta/2\pi$. Since (see [12]) $\delta_0 = 1$ and $\delta_k = 0$ for all $k = 1, 2, \ldots$ it results $\delta_{n+1} = 0$ and $\eta_{n+1} = 1$ for all $n = 0, 1, 2, \ldots$ and hence

$$\mathscr{K}_n(z,\xi) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\xi)} = (-z\bar{\xi})^{d(n)} \frac{\overline{\varphi_n(\xi)}\varphi_n(z) - \overline{\varphi_{n+1}(\xi)}\varphi_{n+1}(z)}{1 - z\bar{\xi}}. \tag{3.18}$$

For this particular measure one can also write

$$\varphi_n(z) = z^{(-1)^n p(n)}, \quad n \geqslant 0$$

and hence an expression for the reproducing kernel function can be obtained from (3.1) or from (3.18), yielding

$$\mathscr{K}_n(z,\xi) = \frac{(z\bar{\xi})^{d(n)-p(n+1)} - (z\bar{\xi})^{d(n)+q(n+1)}}{1 - z\bar{\xi}}, \quad n \geqslant 0,$$

that is,

$$\mathscr{K}_{2k}(z,\xi) = \frac{1 - (z\bar{\xi})^{2k+1}}{(1 - z\bar{\xi})(z\bar{\xi})^k}, \quad \mathscr{K}_{2k+1}(z,\xi) = \frac{1 - (z\bar{\xi})^{2k+2}}{(1 - z\bar{\xi})(z\bar{\xi})^{k+1}}, \quad k \geqslant 0.$$

Example 3.3 (*A rational modification of the Lebesgue measure*). Consider the measure

$$\mathrm{d}\mu(\theta) = \frac{\mathrm{d}\theta}{2\pi |h(\mathrm{e}^{\mathrm{i}\theta})|^2},$$

where $h(z)$ is a polynomial of degree s without zeros on the unit circle $\mathbb{T}$ (if $s = 0$ then $h(z) \equiv 1$ and the normalized Lebesgue measure is recovered). It is known (see [12] and also [8]) that the nth orthonormal Szegö polynomial is given by $\omega_n(z) = z^{n-s}h(z)$ for all $n \geqslant s$, $\delta_s = (-1)^s \prod_{k=1}^{s} \alpha_k$ and $\delta_n = 0$ for all $n > s$. Hence $\eta_{n+1} = 1$, $A_{n+1} = 0$ for all $n \geqslant s$ and the reproducing kernel function is also given by (3.18) for $n \geqslant s$. From (2.3) to (2.4) it holds $\varphi_{2k}(z) = z^{k-s}h(z)$ for $2k \geqslant s$ and $\varphi_{2k+1}(z) = z^{-(k+1)}h^*(z)$ for $2k+1 \geqslant s$. Now we distinguish:

- If $n = 2k \geqslant s$

$$\begin{aligned}(1 - z\bar{\xi})\mathscr{K}_{2k}(z,\xi) &= (-z\bar{\xi})[(z\bar{\xi})^{k-s}h(z)\overline{h(\xi)} - (z\bar{\xi})^{-(k+1)}h^*(z)\overline{h^*(\xi)}] \\ &= (z\bar{\xi})^{-k}h^*(z)\overline{h^*(\xi)} - (z\bar{\xi})^{k+1-s}h(z)\overline{h(\xi)}.\end{aligned} \tag{3.19}$$

- If $n = 2k+1 \geqslant s$

$$(1 - z\bar{\xi})\mathscr{K}_{2k+1}(z,\xi) = (z\bar{\xi})^{-(k+1)}h^*(z)\overline{h^*(\xi)} - (z\bar{\xi})^{k+1-s}h(z)\overline{h(\xi)}. \tag{3.20}$$

From (3.19) and (3.20) we finally obtain

$$\mathscr{K}_n(z,\xi) = \frac{(z\bar{\xi})^{-p(n)}h^*(z)\overline{h^*(\xi)} - (z\bar{\xi})^{p(n+1)-s}h(z)\overline{h(\xi)}}{1 - z\bar{\xi}}. \tag{3.21}$$

When taking $s=1$ and setting $h(z)=(z-r)/(\sqrt{1-r^2})$ with $0<r<1$, a measure associated with the Poisson kernel arises, namely

$$\mathrm{d}\mu(\theta)=\frac{(1-r^2)\mathrm{d}\theta}{2\pi(1-2r\cos\theta+r^2)},\quad r\in(0,1). \tag{3.22}$$

Now, the orthonormal polynomials are given by $\omega_0(z)\equiv 1$ and $\omega_n(z)=(z^n-rz^{n-1})/(\sqrt{1-r^2})$ for $n=1,2,\ldots$. Hence, from (2.3) to (2.4) it results

$$\varphi_{2k}(z)=\frac{z^{n-k}-rz^{n-k-1}}{\sqrt{1-r^2}}\ (k\geqslant 1),\quad \varphi_{2k+1}(z)=\frac{z^{-(k+1)}-rz^{-k}}{\sqrt{1-r^2}}\ (k\geqslant 0),$$

and then from (3.18) (or from (3.21)) it holds $\mathscr{K}_0(z,\xi)\equiv 1$ and

$$\mathscr{K}_n(z,\xi)=\frac{(z\bar{\xi})^{p(n)}L(r)-(z\bar{\xi})^{-p(n)}L^*(r)}{(1-r^2)(1-z\bar{\xi})},\quad n\geqslant 1$$

being $L(r)=-r^2+r(z+\bar{\xi})-(z\bar{\xi})$ (z and ξ parameters).

This section is concluded with a converse result for Theorem 3.1.

Theorem 3.4. *Let $\{\varphi_k(z)\}_{k=0}^{\infty}$ be a family of Laurent polynomials induced by the ordering $p(n)=E[(n+1)/2]$, i.e., for $n=0,1,\ldots$, $\varphi_n\in\mathscr{L}_n\backslash\mathscr{L}_{n-1}$ with $\mathscr{L}_n=span\{z^j:-p(n)\leqslant j\leqslant q(n)\}$, $q(n)=n-p(n)$, satisfying relations* (3.1) *and* (3.3) *where, for all $n=0,1,2,\ldots$, $\{\delta_n\}_{n=0}^{\infty}$ is a complex sequence such that $|\delta_n|<1$, $\{d(n)\}_{n=0}^{\infty}$ and $\{A_n\}_{n=0}^{\infty}$ two sequences given by* (3.4) *and $\{\eta_n\}_{n=0}^{\infty}$ a real sequence given by* (2.5). *Then the family $\{\varphi_k(z)\}_{k=0}^{\infty}$ satisfies the three-term recurrence relation* (2.10).

Proof. From the relation $\varphi_{n+1}(z)\overline{\varphi_{n+1}(\xi)}=\mathscr{K}_{n+1}(z,\xi)-\mathscr{K}_n(z,\xi)$ it follows

$$\begin{aligned}(1-z\bar{\xi})\varphi_{n+1}(z)\overline{\varphi_{n+1}(\xi)}=&\ \eta_{n+2}(-z\bar{\xi})^{d(n+1)}[(\eta_{n+2}\overline{\varphi_{n+1}(\xi)}+A_{n+2}\overline{\varphi_{n+2}(\xi)})\varphi_{n+1}(z)\\&+(\overline{A_{n+2}\varphi_{n+1}(\xi)}-\eta_{n+2}\overline{\varphi_{n+2}(\xi)})\varphi_{n+2}(z)]-\eta_{n+1}(-z\bar{\xi})^{d(n)}\\&\times[(\eta_{n+1}\overline{\varphi_n(\xi)}+A_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_n(z)\\&+(\overline{A_{n+1}\varphi_n(\xi)}-\eta_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_{n+1}(z)],\end{aligned}$$

that is

$$\begin{aligned}(1-z\bar{\xi})\varphi_{n+1}(z)\overline{\varphi_{n+1}(\xi)}=&\ \eta_{n+2}(-z\bar{\xi})^{d(n+1)}(\overline{A_{n+2}\varphi_{n+1}(\xi)}-\eta_{n+2}\overline{\varphi_{n+2}(\xi)})\varphi_{n+2}(z)\\&+[\eta_{n+2}(-z\bar{\xi})^{d(n+1)}(\eta_{n+2}\overline{\varphi_{n+1}(\xi)}+A_{n+2}\overline{\varphi_{n+2}(\xi)})\\&-\eta_{n+1}(-z\bar{\xi})^{d(n)}(\overline{A_{n+1}\varphi_n(\xi)}-\eta_{n+1}\overline{\varphi_{n+1}(\xi)})]\varphi_{n+1}(z)\\&-\eta_{n+1}(-z\bar{\xi})^{d(n)}(\eta_{n+1}\overline{\varphi_n(\xi)}+A_{n+1}\overline{\varphi_{n+1}(\xi)})\varphi_n(z).\end{aligned} \tag{3.23}$$

Assume first that n is odd, i.e., $n=2k+1$. If we compare the coefficients monomials z^{-k} in both sides of Eq. (3.23) and from (3.4) it holds that

$$\begin{aligned}\gamma_{2k}\delta_{2k}\overline{\varphi_{2k}(\xi)}=&-\eta_{2k+1}\gamma_{2k+1}\bar{\xi}(\delta_{2k+1}\overline{\varphi_{2k}(\xi)}-\eta_{2k+1}\overline{\varphi_{2k+1}(\xi)})-\eta_{2k}\gamma_{2k}\delta_{2k}\\&\times(\overline{\delta_{2k}\varphi_{2k-1}(\xi)}-\eta_{2k}\overline{\varphi_{2k}(\xi)})-\eta_{2k}\gamma_{2k-1}(\eta_{2k}\overline{\varphi_{2k-1}(\xi)}+\delta_{2k}\overline{\varphi_{2k}(\xi)}).\end{aligned}$$

Changing the variable ξ by z, dividing both sides by $\gamma_{2k+1} \neq 0$ and from (2.6) it follows

$$\begin{aligned}\eta_{2k+1}\delta_{2k}\overline{\varphi_{2k}(z)} = & -\eta_{2k+1}\delta_{2k+1}\overline{z\varphi_{2k}(z)} + \eta_{2k+1}^2\overline{z\varphi_{2k+1}(z)} - \eta_{2k+1}\eta_{2k}|\delta_{2k}|^2\overline{\varphi_{2k-1}(z)} \\ & + \eta_{2k}^2\eta_{2k+1}\delta_{2k}\overline{\varphi_{2k}(z)} - \eta_{2k}^3\eta_{2k+1}\overline{\varphi_{2k-1}(z)} - \eta_{2k}^2\eta_{2k+1}\delta_{2k}\overline{\varphi_{2k}(z)}.\end{aligned}$$

From definition (2.5) and dividing by $\eta_{2k+1} \neq 0$ it now holds that

$$-\delta_{2k+1}\overline{z\varphi_{2k}(z)} + \eta_{2k+1}\overline{z\varphi_{2k+1}(z)} - \eta_{2k}\overline{\varphi_{2k-1}(z)} - \delta_{2k}\overline{\varphi_{2k}(z)} = 0.$$

Finally, taking conjugates this equation implies (2.12).

The case n even is similarly treated yielding

$$\overline{\delta_{2k+2}\varphi_{2k+1}(z)} - \eta_{2k+2}\overline{\varphi_{2k+2}(z)} + \eta_{2k+1}\overline{z\varphi_{2k}(z)} + \overline{\delta_{2k+1}z\varphi_{2k+1}(z)} = 0.$$

Thus, taking again conjugates (2.11) is obtained and proof now follows by Remark 2.4. □

4. Favard's theorem

In this section we prove a Favard-type theorem for the monic sequence of Laurent polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ by following rather closely the techniques introduced in the Chihara's book [3]. For a proof of Favard's theorem on the unit circle, see e.g. [9] and also [5], where an alternative simpler approach is used (see also [1]). For this purpose, we start by briefly recalling the concept of orthogonality with respect to a Hermitian linear functional. Indeed, let $\{\mu_n\}_{n=-\infty}^{\infty}$ be a complex sequence satisfying $\mu_n = \overline{\mu_{-n}}$ for all $n = 0, 1, 2, \ldots$ (Hermitian) and denote by μ the linear functional defined on Δ by

$$\mu\left(\sum_{j=p}^{q} \alpha_j z^j\right) := \sum_{j=p}^{q} \alpha_j \mu_{-j}, \quad \alpha_j \in \mathbb{C} \quad -\infty < p \leqslant q < +\infty.$$

In terms of μ we define a bilinear functional $\langle \cdot, \cdot \rangle$ on $\Delta \times \Delta$ by

$$\langle L, M \rangle_\mu := \mu(L(z)\overline{M(1/\bar{z})}), \quad L, M \in \Delta. \tag{4.1}$$

Now (see e.g. [9]), it will be said that the functional μ is quasi-definite if and only if the principal submatrices of the infinite Toeplitz moment matrix associated with $\{\mu_n\}_{-\infty}^{\infty}$ are nonsingular and positive definite if the determinants of these matrices are positive. Quasi-definiteness is a necessary and sufficient condition for the existence of a family of orthogonal Laurent polynomials with respect to the linear functional (4.1) in the sense that there exists a sequence $\{R_n(z)\}_{n=0}^{\infty}$ of Laurent polynomials satisfying $R_n(z) \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$, $n = 1, 2, \ldots$ and $\langle R_n(z), R_m(z) \rangle_\mu = k_n \delta_{n,m}$, $k_n \neq 0$. On the other hand, if the linear functional μ is positive-definite then, the associated linear functional (4.1) is an inner product on $\Delta \times \Delta$ and it holds $\langle R_n(z), R_m(z) \rangle_\mu = k_n \delta_{n,m}$ with $k_n > 0$. When $k_n = 1$ for all $n = 0, 1, 2, \ldots$ $\{R_n(z)\}_{n=0}^{\infty}$ will be called "orthonormal".

Theorem 4.1 (*Favard*). *Let $\{\phi_n(z)\}_{n=0}^{\infty}$ be a sequence of Laurent polynomials defined by the recurrence relation*

$$\phi_{2n}(z) = (\delta_{2n} + \delta_{2n-1}z)\phi_{2n-1}(z) + \lambda_{2n-1}z\phi_{2n-2}(z), \quad n \geqslant 1, \tag{4.2}$$

$$\phi_{2n+1}(z) = \left(\overline{\delta_{2n+1}} + \frac{\overline{\delta_{2n}}}{z}\right)\phi_{2n}(z) + \frac{\lambda_{2n}}{z}\,\phi_{2n-1}(z), \quad n \geqslant 1 \tag{4.3}$$

with the initial conditions

$$\phi_0(z) \equiv 1, \quad \phi_1(z) = \overline{\delta_1} + \frac{1}{z}, \tag{4.4}$$

where $\{\delta_n\}_{n=0}^{\infty}$ is a given sequence of complex numbers ($|\delta_n| \neq 1$ for all $n = 1, 2, \ldots$) and $\lambda_n = 1 - |\delta_n|^2$. Then, for a fixed $\mu_0 \in \mathbb{R}\backslash\{0\}$ there exists a unique quasi-definite linear functional μ such that $\mu(1) = \mu_0$ and $\{\phi_n(z)\}_{n=0}^{\infty}$ is the sequence of monic orthogonal Laurent polynomials with respect to μ and the ordering induced in Λ by the sequences $p(n) = E[(n+1)/2]$ and $q(n) = n - p(n) = E[n/2]$. Furthermore, if we take $\mu_0 > 0$ then μ is positive definite if and only if $|\delta_n| < 1$ for all $n = 1, 2, \ldots$.

Proof. We start constructing a sequence of complex numbers $\{\mu_n\}_{n=-\infty}^{\infty}$ satisfying $\mu_m = \overline{\mu_{-m}}$ for all $m = 0, 1, 2, \ldots$ such that the linear functional $\mu^{(n)}$ defined on $\Lambda_{-p(n),q(n)}$ for $n = 1, 2, \ldots$ by

$$\mu^{(n)}\left(\sum_{k=-p(n)}^{q(n)} c_k z^k\right) := \sum_{k=-p(n)}^{q(n)} c_k \mu_{-k}, \quad c_k \in \mathbb{C} \text{ for all } k = -p(n), \ldots, q(n) \tag{4.5}$$

satisfies

$$\mu^{(n)}(\phi_k(z)) = 0, \quad \forall k = 1, 2, \ldots, n. \tag{4.6}$$

Take $\mu_0 \in \mathbb{R}\backslash\{0\}$ fixed and then proceed by induction. Since $\mu_0 = \overline{\mu_0}$, $\mu^{(0)}$ defined by (4.5) satisfies (4.6) trivially. It is also easy to check from (4.2) and (4.4) that setting $\mu_1 = -\overline{\delta_1}\mu_0$ and $\mu_{-1} = \overline{\mu_1}$, (4.6) holds for $n = 1$. Suppose now that for some $n > 1$

$$\mu_{-p(n)}, \ldots, \mu_0, \ldots, \mu_{p(n)} \quad \text{if } n \text{ is even}$$

or

$$\mu_{1-p(n)}, \ldots, \mu_0, \ldots, \mu_{p(n)} \quad \text{if } n \text{ is odd}$$

have been determined such that $\mu^{(n)}$ defined on $\Lambda_{-p(n),q(n)}$ satisfies (4.6). So:

- If n is even then we can write

$$\frac{1}{z^{p(n+1)}} = \left(\frac{1}{z^{p(n+1)}} - \phi_{n+1}(z)\right) + \phi_{n+1}(z) = \phi_{n+1}(z) + \sum_{k=0}^{n} a_k^{(n)}\phi_k(z)$$

with $a_k^{(n)}$ uniquely determined for $k = 0, 1, \dots, n$. Taking $\mu_{p(n+1)} = a_0^{(n)}\mu_0$ then

$$\mu^{(n+1)}(\phi_{n+1}(z)) = \mu^{(n+1)}\left(\frac{1}{z^{p(n+1)}} - \sum_{k=0}^{n} a_k^{(n)}\phi_k(z)\right) = \mu_{p(n+1)} - a_0^{(n)}\mu_0 = 0.$$

- If n is odd then we can write

$$z^{p(n)} = (z^{p(n)} - \phi_{n+1}(z)) + \phi_{n+1}(z) = \phi_{n+1}(z) + \sum_{k=0}^{n} b_k^{(n)}\phi_k(z)$$

with $b_k^{(n)}$ uniquely determined for $k = 0, 1, \dots, n$. Taking now $\mu_{-p(n)} = b_0^{(n)}\mu_0$ then

$$\mu^{(n+1)}(\phi_{n+1}(z)) = \mu^{(n+1)}\left(z^{p(n)} - \sum_{k=0}^{n} b_k^{(n)}\phi_k(z)\right) = \mu_{-p(n)} - b_0^{(n)}\mu_0 = 0.$$

In both situations, since $\mu^{(n+1)}$ is an extension of $\mu^{(n)}$ we also have $\mu^{(n+1)}(\phi_k(z))=0$ for $k=1, 2, \dots, n$. The linear functional μ defined by

$$\mu\left(\sum_{m=p}^{q} c_m z^m\right) := \sum_{m=p}^{q} c_m \mu_{-m}, \quad c_m \in \mathbb{C}, \quad -\infty < p \leqslant q < +\infty \tag{4.7}$$

is an extension of $\mu^{(n)}$ for all $n=0, 1, 2, \dots$ so $\mu(\phi_k(z))=0$ for all $k=1, 2, \dots$. If we define the functional $\langle \cdot, \cdot \rangle$ by

$$\langle X(z), Y(z)\rangle := \mu\left(X(z)\overline{Y}\left(\frac{1}{z}\right)\right) \quad \text{for all } X, Y \in \Lambda \tag{4.8}$$

then it remains to check for $n = 1, 2, \dots$ the orthogonality conditions,

$$\langle \phi_{2n}(z), z^m\rangle = 0, \quad -n \leqslant m \leqslant n-1, \tag{4.9}$$

$$\langle \phi_{2n}(z), z^n\rangle \neq 0, \tag{4.10}$$

$$\langle \phi_{2n+1}(z), z^m\rangle = 0, \quad -n \leqslant m \leqslant n, \tag{4.11}$$

$$\left\langle \phi_{2n+1}(z), \frac{1}{z^{n+1}}\right\rangle \neq 0. \tag{4.12}$$

The condition $\mu_k = \overline{\mu_{-k}}$ for all $k=0, 1, 2, \dots$ (equivalent to show $a_0^{(k)} = \overline{b_0^{(k+1)}}$) holds since $\mu_k = \mu(1/z^k) = \langle 1/z^k, 1\rangle = \overline{\langle 1, z^k\rangle} = \overline{\mu_{-k}}$. Relations (4.9) and (4.11) are valid when $m=0$ since $\langle \phi_k(z), 1\rangle = \mu(\phi_k(z)) = 0$ for $k=1, 2, \dots$ and $\langle \phi_0(z), 1\rangle = \langle 1, 1\rangle = \mu(1) = \mu_0 \in \mathbb{R}\backslash\{0\}$. In order to state (4.9) and (4.11) we separate in two cases:

Case 1:

$$\langle \phi_{2n}(z), z^q\rangle = 0 \quad q = 0, 1, \dots, n-1$$
$$\langle \phi_{2n+1}(z), z^q\rangle = 0 \quad q = 0, 1, \dots, n$$

Case 2:

$$\left\langle \phi_{2n}(z), \frac{1}{z^q} \right\rangle = 0 \quad q = 0, 1, \ldots, n$$
$$\left\langle \phi_{2n+1}(z), \frac{1}{z^q} \right\rangle = 0 \quad q = 0, 1, \ldots, n$$

We shall only prove Case 1 since Case 2 is similarly treated. Indeed, for $r = 0, 1, 2, \ldots$ we define the statements

$$\langle \phi_{2n+1}(z), z^q \rangle = 0, \quad 0 \leqslant q \leqslant r, \quad n = r, r+1, r+2, \ldots, \qquad (I_r)$$

$$\langle \phi_{2n}(z), z^q \rangle = 0, \quad 0 \leqslant q \leqslant r-1, \quad n = r, r+1, r+2, \ldots \qquad (J_r)$$

and prove by induction that both statements are valid for all r. For $r = 0$ (I_0) is valid and (J_0) is empty so both hold. We assume that (I_r) and (J_r) are valid for some $r > 0$. For (J_{r+1}) we also have to check

$$\langle \phi_{2n}(z), z^q \rangle = 0, \quad 0 \leqslant q \leqslant r, \quad n = r+1, r+2, \ldots$$

that is,

$$\langle \phi_{2n}(z), z^r \rangle = 0, \quad n = r+1, r+2, \ldots .$$

From (4.2) we deduce

$$\langle \phi_{2n}(z), z^r \rangle = \delta_{2n} \langle \phi_{2n-1}(z), z^r \rangle + \delta_{2n-1} \langle \phi_{2n-1}(z), z^{r-1} \rangle + \lambda_{2n-1} \langle \phi_{2n-2}(z), z^{r-1} \rangle = 0$$

since $\langle \phi_{2n-1}(z), z^r \rangle = \langle \phi_{2n-1}(z), z^{r-1} \rangle = 0$ from (I_r) and $\langle \phi_{2n-2}(z), z^{r-1} \rangle = 0$ from (J_r). For (I_{r+1}) we have to check

$$\langle \phi_{2n+1}(z), z^q \rangle = 0, \quad 0 \leqslant q \leqslant r+1, \quad n = r+1, r+2, \ldots$$

that is,

$$\langle \phi_{2n+1}(z), z^{r+1} \rangle = 0, \quad n = r+1, r+2, \ldots .$$

From (4.3) we have

$$\phi_{2n+1}(z) = \frac{1}{\lambda_{2n+2}} [z\phi_{2n+3}(z) - (z\overline{\delta_{2n+3}} + \overline{\delta_{2n+2}})\phi_{2n+2}(z)]$$

so

$$\begin{aligned} \langle \phi_{2n+1}(z), z^{r+1} \rangle &= \frac{1}{\lambda_{2n+2}} [\langle \phi_{2n+3}(z), z^r \rangle - \overline{\delta_{2n+3}} \langle \phi_{2n+2}(z), z^r \rangle - \overline{\delta_{2n+2}} \langle \phi_{2n+2}(z), z^{r+1} \rangle] \\ &= \frac{-\overline{\delta_{2n+2}}}{\lambda_{2n+2}} \langle \phi_{2n+2}(z), z^{r+1} \rangle, \end{aligned}$$

since $\langle \phi_{2n+3}(z), z^r \rangle = 0$ from (I_r) and $\langle \phi_{2n+2}(z), z^r \rangle = 0$ from (J_{r+1}). Then, from (4.2),

$$\begin{aligned}
&\langle \phi_{2n+1}(z), z^{r+1} \rangle \\
&\quad = \frac{-\overline{\delta_{2n+2}}}{\lambda_{2n+2}}[\delta_{2n+2}\langle \phi_{2n+1}(z), z^{r+1} \rangle + \delta_{2n+1}\langle \phi_{2n+1}(z), z^r \rangle + \lambda_{2n+1}\langle \phi_{2n}(z), z^r \rangle] \\
&\quad = \frac{-|\delta_{2n+2}|^2}{\lambda_{2n+2}}\langle \phi_{2n+1}(z), z^{r+1} \rangle,
\end{aligned}$$

since $\langle \phi_{2n+1}(z), z^r \rangle = 0$ from (I_r) and $\langle \phi_{2n}(z), z^r \rangle = 0$ from (J_{r+1}). We conclude

$$\langle \phi_{2n+1}(z), z^{r+1} \rangle = \frac{-|\delta_{2n+2}|^2}{\lambda_{2n+2}}\langle \phi_{2n+1}(z), z^{r+1} \rangle$$

and since $-|\delta_{2n+2}|^2/\lambda_{2n+2} = -|\delta_{2n+2}|^2/(1-|\delta_{2n+2}|^2) \neq 1$ there must hold $\langle \phi_{2n+1}(z), z^{r+1} \rangle = 0$.

Once (4.9) and (4.11) have been proved, it remains to check the orthogonality relations (4.10) and (4.12). Indeed, from (4.2) we deduce

$$\langle \phi_{2n}(z), z^n \rangle = \delta_{2n}\langle \phi_{2n-1}(z), z^n \rangle + \lambda_{2n-1}\langle \phi_{2n-2}(z), z^{n-1} \rangle, \tag{4.13}$$

since $\langle \phi_{2n-1}(z), z^{n-1} \rangle = 0$. From (4.3) we also have

$$\phi_{2n-1}(z) = \frac{1}{\lambda_{2n}}[z\phi_{2n+1}(z) - (z\overline{\delta_{2n+1}} + \overline{\delta_{2n}})\phi_{2n}(z)]$$

so

$$\begin{aligned}
\langle \phi_{2n-1}(z), z^n \rangle &= \frac{1}{\lambda_{2n}}[\langle \phi_{2n+1}(z), z^{n-1} \rangle - \overline{\delta_{2n+1}}\langle \phi_{2n}(z), z^{n-1} \rangle - \overline{\delta_{2n}}\langle \phi_{2n}(z), z^n \rangle] \\
&= -\frac{\overline{\delta_{2n}}}{\lambda_{2n}}\langle \phi_{2n}(z), z^n \rangle,
\end{aligned}$$

since $\langle \phi_{2n+1}(z), z^{n-1} \rangle = \langle \phi_{2n}(z), z^{n-1} \rangle = 0$. Now from (4.13)

$$\langle \phi_{2n}(z), z^n \rangle = -\frac{|\delta_{2n}|^2}{\lambda_{2n}}\langle \phi_{2n}(z), z^n \rangle + \lambda_{2n-1}\langle \phi_{2n-2}(z), z^{n-1} \rangle$$

which implies

$$\left(1 + \frac{|\delta_{2n}|^2}{\lambda_{2n}}\right)\langle \phi_{2n}(z), z^n \rangle = \lambda_{2n-1}\langle \phi_{2n-2}(z), z^{n-1} \rangle$$

and so

$$\langle \phi_{2n}(z), z^n \rangle = \lambda_{2n}\lambda_{2n-1}\langle \phi_{2n-2}(z), z^{n-1} \rangle.$$

Continuing in this manner, we obtain

$$\langle \phi_{2n}(z), z^n \rangle = \left(\prod_{k=1}^{2n} \lambda_k\right)\mu_0 = \prod_{k=1}^{2n}(1-|\delta_k|^2)\mu_0 \neq 0$$

and (4.10) follows. We can proceed in a similar way to prove

$$\left\langle \phi_{2n+1}(z), \frac{1}{z^{n+1}} \right\rangle = \left(\prod_{k=1}^{2n+1} \lambda_k \right) \mu_0 = \prod_{k=1}^{2n+1} (1 - |\delta_k|^2)\mu_0 \neq 0.$$

In short, we can write

$$\langle \phi_n(z), \phi_n(z) \rangle = \prod_{k=1}^{n} (1 - |\delta_k|^2)\mu_0 \neq 0, \quad n = 0, 1, 2, \ldots \tag{4.14}$$

and since $\{\phi_k(z)\}_{k=0}^{\infty}$ is a basis of Λ, we see that μ is quasi-definite.

For the proof of the uniqueness, let us assume that there exists another linear functional $\tilde{\mu}$ such that $\{\phi_n(z)\}_{n=0}^{\infty}$ given by (4.2) and (4.3) represents the sequence of monic orthogonal Laurent polynomials for this functional and that $\tilde{\mu}(1) = \mu_0$. We must show that $\mu = \tilde{\mu}$, i.e., $\mu(R) = \tilde{\mu}(R)$ for all $R \in \Lambda$. Now, $\{\phi_n(z)\}_{n=0}^{\infty}$ is a basis of Λ, and because of the orthogonality, for $k \geqslant 1$

$$\mu(\phi_k(z)) = \langle \phi_k(z), 1 \rangle_{\mu} = 0 = \langle \phi_k(z), 1 \rangle_{\tilde{\mu}} = \tilde{\mu}(\phi_k(z)).$$

On the other hand, $\mu(1) = \mu_0 = \tilde{\mu}(1)$ and the uniqueness follows. Finally, positive definiteness follows in a straightforward way from (4.14). □

Remark 4.2. From Lemma 2.1 together with (2.2) and (2.9) one clearly sees that the functional μ is positive-definite if and only if $\Gamma_n > 0$ for all $n = 0, 1, 2, \ldots$.

Now then from the combination of Theorems 3.4 and 4.1 we get.

Corollary 4.3. *Let* $\{\varphi_k(z)\}_{k=0}^{\infty}$ *denote a family of Laurent polynomials induced by the ordering* $p(n) = E[(n+1)/2]$, *satisfying the relations* (3.1) *and* (3.3) *where, for all* $n = 0, 1, 2, \ldots$, $\{\delta_n\}_{n=0}^{\infty}$ *is a complex sequence such that* $|\delta_n| < 1, n = 1, 2, \ldots$, $\{d(n)\}_{n=0}^{\infty}$ *and* $\{A_n\}_{n=0}^{\infty}$ *two sequences given by* (3.4) *and* $\{\eta_n\}_{n=0}^{\infty}$ *a real sequence given by* (2.5). *Then there exists a unique positive measure* μ *such that* $\{\varphi_n(z)\}_{n=0}^{\infty}$ *is the corresponding sequence of orthonormal Laurent polynomials with respect to* μ.

Remark 4.4. Finally it should be observed that if we proceed as in Theorem 4.1, from the recurrence relations (2.14)–(2.15) we can also deduce a Favard's theorem for the family of monic orthogonal polynomials $\{\tilde{\phi}_n(z)\}_{n=0}^{\infty}$ and a new quasi-definite linear functional $\tilde{\mu}$ appears. By assuming that $\mu(1) = \tilde{\mu}(1) = \mu_0 \neq 0$ and since it holds,

$$\tilde{\mu}(\phi_n(z)) = \tilde{\mu}(\tilde{\phi}_{n*}(z)) = \tilde{\mu}(1\overline{\tilde{\phi}_n(1/\bar{z})}) = \langle 1, \tilde{\phi}_n(z) \rangle_{\tilde{\mu}} = 0 = \mu(\phi_n(z)), \quad n = 1, 2, 3, \ldots,$$

it follows $\tilde{\mu} = \mu$. Recall that both $\{\phi_n(z)\}_{n=0}^{\infty}$ and $\{\tilde{\phi}_n(z)\}_{n=0}^{\infty}$ are basis for Λ.

Acknowledgements

The authors thank the referees for their helpful comments and criticism.

References

[1] R. Álvarez-Nodarse, F. Marcellán, On the "Favard's theorem" and its extensions, J. Comput. Appl. Math. 127 (2001) 231–254.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Orthogonal rational functions, Cambridge Monographs on Applied and Computational Mathematics, vol. 5, Cambridge University Press, Cambridge, 1999.

[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[4] R. Cruz-Barroso, P. González-Vera, Orthogonal Laurent polynomials and quadratures on the unit circle and the real half-line, Electron. Trans. Numer. Anal., 2004, to appear.

[5] T. Erdelyi, P. Nevai, J. Ahang, J. Geronimo, A simple proof of the "Favard's theorem" on the unit circle, Atti Sem. Mat. Fis. Univ. Moderna 39 (1991) 551–556.

[6] Ya.L. Geronimus, Polynomials Orthogonal on a Circle and on an Interval, International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, Oxford, 1960.

[7] Ya.L. Geronimus, Polynomials Orthogonal on a Circle and their Applications, American Mathematical Society Translation Series 1, vol. 3, Amer. Math. Soc., Providence, RI, 1962, pp. 1–78.

[8] P. González-Vera, J.C. Santos-León, O. Njåstad, Some results about numerical quadrature on the unit circle, Adv. Comput. Math. 5 (1996) 297–328.

[9] W.B. Jones, O. Njåstad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. London Math. Soc. 21 (1989) 113–152.

[10] O. Njåstad, W.J. Thron, The theory of sequences of orthogonal L-polynomials, in: H. Waadeland, H. Wallin (Eds.), Padé Approximants and Continued Fractions, Det Kongelige Norske Videnskabers Selskab Skrifter, 1983, pp. 54–91.

[11] G. Szegö, On bi-orthogonal systems of trigonometric polynomials, Magyar Tud. Alcad. Kutató Int. Közl 8 (1963) 255–273.

[12] G. Szegö, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[13] W.J. Thron, L-polynomials orthogonal on the unit circle, in: A. Cuyt (Ed.), Nonlinear Methods and Rational Approximation, D. Reidel Publishing Company, Dordrecht, 1988, pp. 271–278.

[14] W. Van Assche, Analytic Aspects of Orthogonal Polynomials, Manuscript, Katholieke Universiteit Leuven, 1993.

[15] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, American Mathematical Society Colloquium Publications, vol. 20, Amer. Math. Soc., Providence, RI, 1969.